MOTIVIC DECOMPOSITION FOR RELATIVE GEOMETRICALLY CELLULAR STACKS

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ABSTRACT. We generalize a motivic decomposition theorem due to Karpenko to relative geometrically cellular Deligne-Mumford stacks. Our proof is different from Karpenko's as it relies on a vanishing result of Voevodsky. Even in the classical case, our method yields a simpler and more conceptual proof of Karpenko's result.

1. Introduction

Recall that a *relative cellular variety* is a smooth and proper variety equipped with an increasing sequence of closed subvarieties whose successive differences, called *cells*, are affine fibrations over proper varieties, called *bases*. By a result of Karpenko [Kar, Corollary 6.11], the Chow motive of a relative cellular variety decomposes into the direct sum of the Chow motives of the bases suitably shifted and twisted. In the case of rational coefficients, this was independently shown by del Baño [Ban, Theorem 2.4].

In this paper, we introduce a notion of relative cellularity for Deligne-Mumford stacks and generalize the decomposition of Karpenko. Even for varieties, our notion of relative cellularity is more general than the previous one: instead of asking that the fibers of the map from a cell to its base are affine spaces, we only ask so for the *geometric fibers*. We term this property: *relative geometrically cellular*. The price to pay for such a generality is that our decomposition holds only with rational coefficients. Also, we stress that our proof restricted to varieties simplifies those of Karpenko and del Baño by applying a vanishing theorem of Voevodsky [Voe2, Corollary 4.2.6]. This vanishing theorem gives a simpler proof that the relevant Gysin sequence splits.

Remark 1.1. Let k be a field of characteristic zero, fixed throughout. This is the setting in which the vanishing lemma of Voevodsky (4.1)

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and the result that Chow motives embed in Voevodsky's category of motives are only known to hold. If functorial resolution of singularities were available in positive characteristic, then the arguments stated here could be extended to Deligne-Mumford stacks over any perfect field. Even without functorial resolution of singularities, if the program mentioned in [Kel, section 1.6] is successful, the results hold in positive characteristic $p = \mathrm{char}\ k$ with coefficients in $\mathbb{Z}[1/p]$.

Remark 1.2. In what follows, arguments about motives of Deligne-Mumford stacks in $\mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k, \mathbf{Z})$ remain true in $\mathbf{DM}^{\mathit{eff}}(k, \mathbf{Q})$.

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2. Quick review of motives

Let R be a commutative ring. As usual, we denote by $\mathbf{DM}^{\mathit{eff}}(k, R)$ the triangulated category of effective motives with coefficients in R and by $\mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k, R)$ its étale version, i.e., where the Nisnevich topology is replaced by the stronger étale topology. These categories are constructed in [MVW, Definitions 9.2 and 14.1]. When R is a $\mathbf{Q}$-algebra, there is a canonical equivalence of triangulated categories $\mathbf{DM}^{\mathit{eff}}(k, R) \cong \mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k, R)$ (cf. [MVW, Theorem 19.30]).

Let $SmCor(k)$ be the additive category of finite correspondences. Recall that the objects of $SmCor(k)$ are all smooth k-schemes and that the morphisms between two smooth k-schemes X and Y are given by the free abelian group $Cor(X, Y)$ generated by the integral closed subschemes $W \subset X \times Y$ that are finite and surjective on a connected component of X. Presheaves with transfers are contravariant, additive functors from $SmCor(k)$ to the category of R-modules. They form an abelian category denoted by $PST(k, R)$. For a smooth k-scheme X, the presheaf with transfers $R_{tr}(X)$ is given by $Y \in Sm/k \mapsto Cor(Y, X) \otimes R$.

Objects of $\mathbf{DM}^{\mathit{eff}}(k, R)$ (resp. $\mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k, R)$) are just complexes of presheaves with transfers. However, in this triangulated category there are more isomorphisms than in the derived category of $PST(k, R)$. Indeed, a morphism $f : K' \to K$ between complexes of presheaves with transfers is invertible in $\mathbf{DM}^{\mathit{eff}}(k, R)$ (resp. $\mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k, R)$) if it induces quasi-isomorphisms on the stalks for the Nisnevich (resp. étale) topology. Also, for every smooth k-scheme X, the map $R_{tr}(\mathbf{A}^1 \times X) \to R_{tr}(X)$ is invertible in $\mathbf{DM}^{\mathit{eff}}(k, R)$ (resp. $\mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k, R)$). The *motive* of a smooth k-scheme X, denoted $M(X)$, is the sheaf $R_{tr}(X)$ viewed as an object of $\mathbf{DM}^{\mathit{eff}}(k, R)$ (resp. $\mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k, R)$).

Since k has characteristic zero, there exists a fully faithful functor $\iota : Ch^{\mathit{eff}}(k,R) \to \mathbf{DM}^{\mathit{eff}}(k,R)$ [MVW, Proposition 20.1], where $Ch^{\mathit{eff}}(k,R)$ is the category of effective *Chow motives* with coefficients in R. A motive $M \in \mathbf{DM}^{\mathit{eff}}(k,R)$ is called a *Chow motive* if it is in the essential image of ι.

Let F be a smooth and separated Deligne-Mumford stack over the field k. One can associate to F a motive $M(F) \in \mathbf{DM}^{\mathit{eff}}(k,\mathbf{Q})$ (see [Ch, Definition 2.7]). This motive can be described as follows. Let $U \to F$ be an étale atlas and consider the associated Čech simplicial scheme $U_\bullet$ formed from the $(i+1)$-fold products $U_i = U \times_F \cdots \times_F U$. Relative diagonals and partial projections serve as the face and degeneracy maps. This simplicial scheme determines a complex of presheaves with transfers $N(\mathbf{Q}_{tr}(U_\bullet))$ and hence an object $M(U_\bullet)$ of $\mathbf{DM}^{\mathit{eff}}(k,\mathbf{Q})$. (Here $N(-)$ is the "normalized chain complex" associated to a simplicial group; see [GJ, p. 145].) Then, by [Ch, Corollary 2.14], there is a canonical isomorphism $M(U_\bullet) \cong M(F)$ in $\mathbf{DM}^{\mathit{eff}}(k,\mathbf{Q})$. Similarly, for any coefficient ring R, one can associate to F a motive $M(F) \in \mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k,R)$. By [Ch, Corollary 4.7], it is known that $M(F)$ is a direct factor of the motive of a smooth and quasi-projective k-scheme in $\mathbf{DM}^{\mathit{eff}}(k,\mathbf{Q})$.

3. Affine fibrations

The results on cellular stacks follow in part from the observation that the motive of the total space of a geometric affine fibration is isomorphic to the motive of the base space. The proof proceeds first through the case of schemes.

In this section, the base field k may be a perfect field of any characteristic.

Lemma 3.1 (Homotopy invariance)**.** *Let X be a smooth k-scheme and let $p : Y \to X$ be a* geometric affine fibration*, i.e., a flat morphism whose geometric fibers are affine spaces. Then $M(Y) \cong M(X)$ in $\mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k,\mathbf{Z})$. If moreover all fibers of p are affine spaces, then the isomorphism additionally holds in $\mathbf{DM}^{\mathit{eff}}(k,\mathbf{Z})$.*

Remark 3.1. Under the stronger of the two hypotheses, this lemma generalizes work of Chernousov, Gille, and Merkurjev for varieties [CGM][Theorem 7.2]. We could not find any proof of the stated lemma in the literature, so we include it here.

Proof. Examining the restrictions of the morphism to each connected component, the smooth scheme X may be assumed to be irreducible.

Let n be the relative dimension of p. The smooth scheme X can be filtered by open subschemes,

$$\emptyset = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = X,$$

such that for all i, $W_i := V_i \setminus V_{i-1}$ is smooth, and there is a pullback square of the form,

$$\begin{array}{ccc} \mathbf{A}^n_{U_i} & \longrightarrow & U_i \\ {\scriptstyle \text{ét}}\downarrow & & \downarrow{\scriptstyle \text{ét}} \\ p^{-1}(W_i) & \longrightarrow & W_i \end{array}$$

such that the vertical morphisms are étale and surjective. Indeed, the hypothesis applied to the generic point of X shows that there exists an étale morphism $u_1 : U_1 \to X$ such that $Y \times_X U_1 \cong \mathbf{A}^n_{U_1}$. If u_1 is not surjective, this process may be continued inductively as follows.

Let V_1 be the image of u_1, and let $Z_1 = X \setminus V_1$ be its complement. The field k being of characteristic zero, Z_1 is generically smooth. It has a dense, smooth subscheme $W_2 \subset Z_1$ which is the image of an étale morphism $u_2 : U_2 \to Z_1$ trivializing the affine bundle, i.e., $Y \times_X U_2 \cong \mathbf{A}^n_{U_2}$. Let $V_2 := V_1 \cup W_2$. Since $\operatorname{codim}_X(V_2 \setminus V_1) < \operatorname{codim}_X(V_1 \setminus V_0)$, this process terminates in the promised filtration of X.

We use induction on i to show that $M(V_i) \cong M(p^{-1}(V_i))$ for all i. The case $i = 1$ follows from the observation that $M(V_1) \cong M((U_1)_\bullet)$ and $M(p^{-1}(V_1)) \cong M(p^{-1}(V_1) \times_{V_1} (U_1)_\bullet)$ where $(U_1)_\bullet$ is the Čech simplicial schemes associated to the étale cover $U_1 \to V_1$. In each simplicial degree, $p^{-1}(V_1) \times_{V_1} (U_1)_\bullet$ is isomorphic to $\mathbf{A}^n \times (U_1)_\bullet$. Hence the canonical morphism, $\mathbf{Z}_{tr}(p^{-1}(V_1) \times_{V_1} (U_1)_\bullet) \to \mathbf{Z}_{tr}((U_1)_\bullet)$, induces an $\mathbf{A}^1$-weak equivalence in each simplicial degree. This proves that $M(V_1) \cong M(p^{-1}(V_1))$ in $\mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k, \mathbf{Z})$.

For general i, we use the Gysin triangle from [Voe2, Proposition 3.5.4] (for the étale version of the Gysin triangle, one can use the fact that the canonical functor $\sigma : \mathbf{DM}^{\mathit{eff}}(k, \mathbf{Z}) \to \mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k, \mathbf{Z})$ is a triangulated functor by [MVW, Remark 14.3]). The required isomorphism then follows from the morphism of triangles,

$$\begin{array}{ccccc} M(p^{-1}(V_{i-1})) & \longrightarrow & M(p^{-1}(V_i)) & \longrightarrow & M(p^{-1}(W_i))(c_i)[2c_i] \\ \downarrow\wr & & \downarrow & & \downarrow\wr \\ M(V_{i-1}) & \longrightarrow & M(V_i) & \longrightarrow & M(W_i)(c_i)[2c_i] \end{array}$$

where $c_i := \operatorname{codim}_{V_i} W_i$. □

Remark 3.2. For the Nisnevich topology, we have assumed that *all* fibers are affine. It is not even known whether the lemma holds upon restricting the hypothesis to geometric fibers when p is geometrically an $\mathbb{A}^3$-fibration over a point (see [Kra, Remark 4]).

Corollary 3.2. *Let $p : F' \to F$ be a smooth, representable morphism of smooth Deligne-Mumford stacks such that the geometric fibers are affine spaces. Then $M(F') \cong M(F)$ in $\mathbf{DM}^{\textit{eff}}_{\acute{e}t}(k, \mathbf{Z})$.*

Proof. Fixing an atlas $U \to F$, let $V = U \times_F F' \to F'$ be the induced atlas for F'. Let $U_\bullet$ and $V_\bullet$ be the associated Čech simplicial schemes. There is a natural morphism $V_\bullet \to U_\bullet$, and in each simplicial degree i, $V_i \to U_i$ is a geometric affine fibration. It follows that the morphism $\mathbf{Z}_{tr}(V_\bullet) \to \mathbf{Z}_{tr}(U_\bullet)$ is an $\mathbf{A}^1$-weak equivalence in each simplicial degree. This shows that $M(p) : M(F') \to M(F)$ is an isomorphism. □

4. Motivic decompositions

Remark 4.1. In this section all the results for schemes depends on whether resolution of singularities are available over k. For stacks we need functorial resolution of singularities.

Our motivic decomposition theorem is based on the following lemma which is essentially due to Voevodsky.

Lemma 4.1. *Let $X, Y \in Sm/k$, such that X is proper. Then*

$$Hom_{\mathbf{DM}^{\textit{eff}}(k,\mathbf{Z})}(M(Y)(c)[2c], M(X)[1]) = 0.$$

Proof. When Y is proper, this is [Voe2, Corollary. 4.2.6]. We follow the same argument here. Let $d = \dim(X)$. Since X is proper, by [MVW, Example 20.11] we have

$$\underline{Hom}(M(X), \mathbf{Z}(d)[2d]) \cong M(X).$$

Hence, by [MVW, Proposition 14.16 and Theorem 19.3]

$$\begin{aligned} Hom_{\mathbf{DM}^{\textit{eff}}(k,\mathbf{Z})}(M(Y)(c)[2c], M(X)[1]) &= H_M^{2(d-c)+1}(Y \times X, d-c) \\ &= 0, \end{aligned}$$

where $H^p_M(-, q)$ is motivic cohomology in degree p and weight q (cf. [MVW, Definition 3.4]). □

Proposition 4.2. *Let F be a smooth Deligne-Mumford stack and let $Z \subset F$ be a smooth and closed substack of codimension c. Assume that $M(F \setminus Z)$ is a Chow motive. Then there is an isomorphism in $\mathbf{DM}^{\textit{eff}}(k, \mathbf{Q})$,*

$$M(F) \cong M(Z)(c)[2c] \oplus M(F \setminus Z).$$

If F is a scheme, then the isomorphism holds in $\mathbf{DM}^{\textit{eff}}(k, \mathbf{Z})$.

Proof. By [Ch, Lemma 3.9], there is an exact triangle,

$$M(F \setminus Z) \to M(F) \to M(Z)(c)[2c] \to M(F \setminus Z)[1].$$

To prove this triangle splits, it suffices to show that

$$Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbf{Q})}(M(Z)(c)[2c], M(F \setminus Z)[1]) = 0.$$

This follows from Lemma 4.1, since $M(F \setminus Z)$ is a Chow motive and $M(Z)$ is a direct factor of $M(Y)$ for some smooth k-scheme Y (cf. [Ch, Corollary 4.7]). □

Remark 4.2. Keeping the notation as in Proposition 4.2, let X_i be a smooth and proper Deligne-Mumford stack of pure dimension d_i for $1 \leq i \leq n$, and let $\sigma_i \in Ch^{c_i}((F \setminus Z) \times X_i)$ be a cycle of codimension c_i. Since each X_i is proper, each cycle σ_i induces a morphism

$$\sigma_i : M(F \setminus Z) \to M(X_i)(c_i - d_i)[2(c_i - d_i)]$$

in $\mathbf{DM}^{\mathit{eff}}(k, \mathbf{Q})$ by [Ch, Lemma 4.4] and [J]. Furthermore, assume that the morphism

$$\cup_i \sigma_i : M(F \setminus Z) \to \bigoplus_{i=1}^{n} M(X_i)(c_i - d_i)[2(c_i - d_i)]$$

is an isomorphism in $\mathbf{DM}^{\mathit{eff}}(k, \mathbf{Q})$.

To give an isomorphism as in Proposition 4.2, it is enough to give a splitting of the morphism $\iota : M(F \setminus Z) \to M(F)$. Hence, we need to find cycles $\sigma'_i \in Ch^{c_i}(F \times X_i)$, such that $\sigma'_i \circ \iota = \sigma_i$. The Zariski closures $\sigma'_i := \overline{\sigma_i}$ of σ_i in $F \times X_i$ will suffice and induce an isomorphism,

$$(\cup_i \sigma'_i) \cup \sigma_Z : M(F) \to \left(\bigoplus_{i=1}^{n} M(X_i)(c_i - d_i)[2(c_i - d_i)] \right) \oplus M(Z)(c)[2c],$$

where $\sigma_Z \in Ch^{c+\dim(Z)}(F \times Z)$ is the graph of the inclusion $Z \subset F$.

Definition 4.3. *A* Chow cellular *Deligne-Mumford stack is a smooth Deligne-Mumford stack F endowed with a finite increasing filtration by closed (not necessarily smooth) substacks,*

$$\emptyset = F_{-1} \subset F_0 \subset \cdots \subset F_n = F,$$

such that the successive differences $F_{i \setminus i-1} = F_i \setminus F_{i-1}$, called cells*, are smooth of pure codimension in F, and have Chow motives for motives.*

On the other hand, if each cell $F_{i \setminus i-1}$ admits a geometric affine fibration $F_{i \setminus i-1} \to Y_i$ to a smooth, proper Deligne-Mumford stack Y_i, called the base *of $F_{i \setminus i-1}$, then the stack F is said to be* relative geometrically cellular*. (Compare with* [Kar, Definition 6.1]*).*

Remark 4.3. Relative geometrically cellular Deligne-Mumford stacks are also Chow cellular. Indeed, by Corollary 3.2, we have $M(F_{i\setminus i-1}) \cong M(Y_i)$ for all i and the $M(Y_i)$'s are Chow motives by [Ch, Theorem 3.9].

Proposition 4.4. *Let F be a Chow cellular Deligne-Mumford stack. Then there is an isomorphism,*

$$M(F) \cong \bigoplus_{i=0}^{n} M(F_{i\setminus i-1})(c_i)[2c_i],$$

in $\mathbf{DM}^{eff}(k, \mathbf{Q})$ where $c_i = \operatorname{codim}_F(F_{i\setminus i-1})$. If F is a scheme, the isomorphism holds in $\mathbf{DM}^{eff}(k, Z)$.

Proof. The proof proceeds by descending induction on i applied to the hypothesis that

$$M(F \setminus F_i) \cong \bigoplus_{j=i+1}^{n} M(F_{j\setminus j-1})(c_j)[2c_j].$$

The result will then follow upon reaching $i = -1$.

For $i = n-1$, there is nothing to prove. Now let the result be known for $i+1 \leq n-1$. As the motive $M(F_{i+1\setminus i})$ is Chow by assumption, Proposition 4.2 gives an isomorphism

$$M(F \setminus F_i) \simeq M(F \setminus F_{i+1}) \oplus M(F_{i+1\setminus i})(c_i)[2c_i].$$

The conclusion follows by the induction hypothesis. □

Corollary 4.5. *Let F be a relative geometrically cellular stack, retaining the above notation. Then there is an isomorphism,*

$$M(F) \cong \bigoplus_{i=0}^{n} M(Y_i)(c_i)[2c_i],$$

where $c_i = \operatorname{codim}_F(F_{i\setminus i-1})$.

Proof. The proof follows from Remark 4.3 and Proposition 4.4. □

Remark 4.4. The isomorphism in Corollary 4.5 is induced by the correspondences $\overline{\Gamma}_i \in Ch^{c_i+\dim(Y_i)}(F \times Y_i)$, where $\overline{\Gamma}_i \subset F \times Y_i$ is the closure of the graph of the morphism $F_{i\setminus i-1} \to Y_i$ inside $F \times Y_i$. This follows from Remark 4.2 and the proof of Proposition 4.4.

Remark 4.5. Everything in this section works integrally in the classical case of relative cellular varieties. Hence, it yields a new proof of Karpenko's decomposition theorem [Kar, Corollary 6.11].

Example 4.1. Let X be a smooth and proper Deligne-Mumford stack over an algebraically closed field k of characteristic zero equipped with an action of the multiplicative group $\mathbf{G}_m$. Such an action requires the data of an action map as well as two coherent 2-isomorphisms making relevant diagrams commute [R]. If the coarse moduli space of X is a scheme, then X admits a Białynicki-Birula decomposition [Sko]. More precisely, if $F = \coprod_i F_i$ is the decomposition into connected components of the fixed point locus of the action of $\mathbf{G}_m$, then X decomposes into a disjoint union of locally closed substacks X_i which are $\mathbf{G}_m$-equivariant affine fibrations over the F_i's. It follows that the Chow motive of X in $\mathbf{DM}^{eff}(k, \mathbf{Q})$ decomposes as follows

$$M(X) \cong \bigoplus_i M(F_i)(c_i)[2c_i]$$

where $c_i = \operatorname{codim}_X(X_i)$. For example, this gives decompositions of the motives of spaces of stable maps to $\mathbb{P}^n$ [O] and the moduli space of stable quotients [C].

References

[Ban] S. del Baño, *On the Chow motive of some moduli spaces*, J. Reine Angew. Math. 532(2001), 105-132.

[CGM] V. Chernousov, S. Gille, and A. Merkurjev, *Motivic decomposition of isotropic projective homogeneous varieties*, Duke Math. J. 126, no. 1 (2005), 137-159.

[Ch] U. Choudhury, *Motives of Deligne-Mumford Stacks*, Preprint (2012). arXiv:1109.5288v2 [math.AG].

[C] Y. Cooper, *The Geometry of Stable Quotients in Genus One*, Preprint (2011).

[GJ] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, 174 (1999). Birkhauser.

[J] R. Joshua, *Higher Intersection Theory on Algebraic Stacks, I*, K-Theory, 27, no.2, (2002), 134-195.

[Kar] N. Karpneko, *Cohomology of relative cellular spaces and isotropic flag varieties*, St. Petersburg Math. J. 12 (2001), no. 1, 1-50.

[Kel] S. Kelly, *Vanishing of negative K-theory in positive characteristic*, arXiv:1112.5206v3 [math.AG].

[Kra] H, Kraft: *Challenging problems in affine n-space*, Astérisque, Séminaire Bourbaki, Vol. 1994/95 237, 1996, H. Exp. No. 802, S. 295-317.

[MVW] C. Mazza, V. Voevodsky and C. A. Weibel, *Lecture notes on motivic cohomology*, http://math.rutgers.edu/~weibel/motiviclectures.html.

[O] D. Oprea, .

[R] Romagny, Matthieu, *A note on group actions on algebraic stacks*, Preprint (2003). arXiv:0305243 [math.AG].

[Sko] J. Skowera, *Białynicki-Birula decomposition of Deligne-Mumford stacks*, Proc. Amer. Math. Soc. 141 (2013), 1933-1937.

[Voe1] V. Voevodsky, *Cohomological theory of presheaves with transfers.* Cycles, transfers, and motivic cohomology theories, Annals of Mathematics Studies, vol. 143, Princeton University Press, 2000, p. 87-137.

[Voe2] V. Voevodsky, *Triangulated categories of motives over a field.* Cycles, transfers, and motivic cohomology theories, Annals of Mathematics Studies, vol. 143, Princeton University Press, 2000, p. 188-238.

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