BIAŁYNICKI-BIRULA DECOMPOSITION OF DELIGNE-MUMFORD STACKS

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Abstract. This short note considers the Białynicki-Birula decomposition of Deligne-Mumford stacks under one-dimensional torus actions and extends a result of Oprea.

1. Introduction

In this short note, we consider actions of one-dimensional tori on tame Deligne-Mumford stacks which are smooth and proper over an algebraically closed field. We extend a result of Oprea [Opr06] to show that in the aforementioned case, if the stack has a scheme for a coarse moduli space, or if it is toric, then it admits a Białynicki-Birula decomposition and often a corresponding decomposition of cohomology. A history of the result can be found in [Bro05, Theorem 3.2].

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2. Notation and terminology

Let k be an algebraically closed field of arbitrary characteristic, fixed in what follows.

In this note, an algebraic stack will be a stack X fibered over (Sch/k) in the étale topology, such that the diagonal mapping $\Delta : X \to X \times X$ is representable, separated and quasi-compact, and such that there exists a smooth, surjective k-morphism $U \to X$ from a k-scheme U, which will be called an atlas. Deligne-Mumford stacks are those admitting étale atlases. Proper Deligne-Mumford stacks are those admitting a finite, surjective morphism from a proper k-scheme. Tame Deligne-Mumford stacks are those with linearly reductive geometric stabilizer groups.

An affine fibration is a flat morphism $p : E \to X$ which is étale locally a trivial bundle of affine spaces. This definition weakens the

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definition of a vector bundle by relaxing the requirement that the transition functions be linear.

Let T be a one-dimensional torus over k with an isomorphism to $\boldsymbol{G}_m$. Let the fixed points of an action of T on a stack X be denoted by X^T [Rom05].

If an algebraic stack has the form of a quotient $[X/G]$ of a normal toric variety X over an algebraically closed field of characteristic zero by a subgroup G of the torus of X, then it is called a toric stack [GS11].

3. Białynicki-Birula decomposition of Deligne-Mumford stacks

Oprea proved a form of the Białynicki-Birula decomposition [BB73] for Deligne-Mumford stacks assuming there exists a T-equivariant, affine, étale atlas [Opr06, Proposition 5].

Proposition 3.1 (Oprea)**.** *Let X be a smooth, proper Deligne-Mumford stack over an algebraically closed field k with a T-action that admits a T-equivariant, affine, étale atlas $U \to X$. Let $F = \coprod_i F_i$ be the decomposition of the fixed substack into connected components. Then X decomposes into disjoint, locally closed, T-equivariant substacks X_i which are T-equivariant affine fibrations over F_i.*

Oprea expected the existence of an atlas to be a general fact [Opr06, Section 2]. Here we show that the desired atlas exists under somewhat general conditions.

Proposition 3.2. *Let X be a tame, irreducible Deligne-Mumford stack, smooth and proper over k, whose generic stabilizer is trivial and whose coarse moduli space is a scheme. Furthermore, let an action of T on X be given such that T acts trivially on its fixed locus. Then there exists a T-equivariant, affine, étale atlas $U \to X$.*

Remark 3.3. If an algebraic group G acts algebraically on a Deligne-Mumford stack X, then the action descends to the coarse moduli space of X by the universal property of the coarse moduli space of $G \times_k X$.

Remark 3.4. If T does not act trivially on its fixed locus, it can be made to do so by a reparametrization of the action. Since T-invariance is not affected by this change, the decomposition for the new action will be a decomposition for the original action.

Proof. Since X is reduced and irreducible with a scheme for a coarse moduli space, the Keel-Mori theorem [KM97, Proposition 4.2] shows the coarse moduli space is in fact a normal, proper variety over k. It inherits a T-action by the remark 3.3. Since it is proper, any collection of open, T-invariant neighborhoods containing the fixed point locus covers it and contains a finite subcover. They can be chosen affine by Sumihiro's theorem [Sum74, Corollary 2]. So it suffices to find the desired atlas for an arbitrary fixed k-point $x \in X$. To this end, let Y be the pullback of X to an open, affine, T-invariant neighborhood of the image of x in the coarse moduli space.

Consider the frame bundle of Y with total space FY. Each k-point $y \in Y$ lies in the image of an étale, representable morphism from a quotient stack of the form $[U/G]$ for an affine, irreducible scheme U with an action of the stabilizer group G of y and containing a point u fixed by the G-action and mapping to y [KM97]. By the tameness hypothesis, G is linearly reductive. Since Y has trivial generic stabilizer, G acts faithfully on U. Applying [BB73, Theorem 2.4] to U and T_uU shows that G acts faithfully on T_yY. So the total space FY is an algebraic space, and $Y = [FY/\boldsymbol{GL}_n]$, where n is the dimension of X. But Y has an affine coarse moduli space, so FY is, in fact, an affine scheme (cf. [EHKV01, Remark 4.3]).

The action of T on Y induces an action on TY and hence an action on FY, which is an open, T-invariant substack of $TY^{\oplus n}$. Let $p : FY \to Y$ be the projection. An atlas will be defined by finding a T-equivariant étale slice $U \hookrightarrow FY$ over the fixed point y.

One may assume a fixed point $f \in FY$ lies over y by modifying the action as follows. Choose a basis of the tangent space T_yY which diagonalizes the T-action, i.e., so $t : (v_1, \ldots, v_n) \mapsto (t^{a_1}v_1, \ldots, t^{a_n}v_n)$. Then the induced T^n-action on FY can be used to define a twisted T-action on FY by

$$\begin{array}{ccc} T & \to & T^n \\ t & \mapsto & (t^{-a_1}, \ldots, t^{-a_n}). \end{array}$$

The projection from the frame bundle remains T-equivariant after twisting the action, but now the action fixes the frame f formed by the basis vectors.

The proof finishes by arguing as in [Opr06, Lemma 3]. The torus T acts on the tangent space T_fFY at f, with $T_fFY \to T_yY$ surjective and T-equivariant. By the linear reductivity of T, T_yY may be identified with some T-invariant subspace $N \subset T_yFY$, compatibly with the T-actions. By a theorem of Białynicki-Birula [BB73, Theorem 2.1], there exists a reduced and irreducible, closed, T-invariant subscheme Z of FY containing f as a non-singular point such that $T_fZ = N$.

Taking U to be the largest open subscheme of Z on which the restriction of p is étale ensures that U is T-invariant. By applying Sumihiro's theorem again, one may shrink U to an affine, T-invariant neighborhood of f whose image in Y contains y. □

Theorem 3.5. *Let X be a tame Deligne-Mumford stack, smooth and proper over k, whose coarse moduli space is a scheme, and let an action of T on X be given. Then X admits a Białynicki-Birula decomposition.*

Proof. One first reduces to the case that X is irreducible by decomposing each irreducible component separately and combining to give a decomposition of all of X. Applying Remark 3.4, one may suppose T acts trivially on its fixed locus. By [Ols07, Proposition 2.1], there exists a rigidification $\overline{X}$ of X which has trivial generic stabilizer. In particular, there is an étale, proper morphism $X \to \overline{X}$ which forms a G-gerbe for a finite group G. The T-action descends to $\overline{X}$ using the universal property of rigidification, so Proposition 3.2 guarantees the existence of a T-equivariant, affine, étale atlas of $\overline{X}$. Let the substacks $\overline{F}_i$ and $\overline{X}_i$, together with T-equivariant affine fibrations $\overline{X}_i \to \overline{F}_i$, be defined according to Proposition 3.1.

Let the decomposition of X be defined by pulling back along $X \to \overline{X}$. Pulling back again by the affine fibration $\overline{X}_i \to \overline{F}_i$ forms the diagram:

$$\begin{array}{ccccccc} \overline{X}_i \times_{\overline{F}_i} F_i & \longrightarrow & F_i & \longrightarrow & X_i & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \overline{X}_i & \longrightarrow & \overline{F}_i & \longrightarrow & \overline{X}_i & \longrightarrow & \overline{X} \end{array} \tag{3.1}$$

So $F = \coprod_i F_i$ is a decomposition of the fixed substack into connected components [Rom05]. Furthermore, all morphisms are T-equivariant.

In what follows, let i be fixed. It remains to prove the existence of an affine fibration, and diagram (3.1) shows it will suffice to supply a T-equivariant isomorphism $\overline{X}_i \times_{\overline{F}_i} F_i \to X_i$ over $\overline{X}_i$. This can be done by specifying such an isomorphism, unique up to canonical 2-isomorphism, over an étale, T-equivariant atlas of $\overline{X}_i$, and then applying the descent property of the stack $\underline{\mathrm{Hom}}_{\overline{X}_i}(\overline{X}_i \times_{\overline{F}_i} F_i, X_i)$ [Ols06].

First, étale, T-equivariant atlases forming the following pullback will be defined:

$$\begin{array}{ccc} W & \longrightarrow & P \\ \downarrow & & \downarrow \\ \overline{X}_i & \longrightarrow & \overline{F}_i \end{array} \tag{3.2}$$

There is an $\mathrm{Out}(G)$-torsor on $\overline{F}_i$ associated to the rigidification gerbe $F_i \to \overline{F}_i$ whose class α lies in $H^1(\overline{F}_i, \mathrm{Out}(G))$. Base change to the total space of the torsor trivializes α and gives an element of the étale cohomology $\beta \in H^2(\overline{F}_i, Z)$ which classifies the gerbe, where Z is the center of G (cf. [EHKV01, Section 3.1]). Let $P \to \overline{F}_i$ be an affine, étale atlas that trivializes α, β, and the affine fibration $\overline{X}_i \to \overline{F}_i$. Also, let $W := P \times_{\overline{F}_i} \overline{X}_i$, as in diagram (3.2).

By the Künneth formula for the algebraic fundamental group [Ray71, Proposition 4.6], the atlas W trivializes the $\mathrm{Out}(G)$-torsor associated to the gerbe $X_i \to \overline{X}_i$. Then [Art73, Corollary 2.2] shows that the morphism $P \times \boldsymbol{A}^n \cong W \to P$ induces an isomorphism $H^2(P, Z) \xrightarrow{\sim} H^2(W, Z)$, implying that the classifying element of $X_i \times_{\overline{X}_i} W \to W$ vanishes. The isomorphism $X_i \times_{\overline{X_i}} W \xrightarrow{\sim} F_i \times_{\overline{F}_i} W$ of trivial gerbes can be chosen, uniquely up to canonical 2-isomorphism, to be the isomorphism over W which extends the identity morphism of $F_i \times_{\overline{F}_i} P$ over P. This follows from the triviality of the affine fibration $W \to P$ and the Künneth formula. The isomorphism will also be T-equivariant by similar reasoning, since the identity is T-equivariant. ☐

Remark 3.6. If X is a tame Deligne-Mumford stack, smooth and proper over k, with a projective coarse moduli space and a T-action, then the induced decomposition forms a filtration, and a lemma of Oprea [Opr06, Lemma 6] implies that the Betti numbers of the stack are calculated by the Betti numbers of the fixed points.

In what follows, T may be a torus of arbitrary dimension.

Proposition 3.7. *Let* char $k = 0$*, and let X be a normal algebraic space, separated and of finite type over k, with an action of T which gives a dense, open embedding of T in X. Then X is a scheme, and hence a toric variety.*

Proof. First, let $k = \boldsymbol{C}$. The scheme locus of the normalized blow-up at the closure of any non-dense T-orbit forms a toric variety whose image includes the orbit. The associated fans give, in each T-orbit, a limit point of a $\boldsymbol{G}_m$-orbit of $1 \in T \hookrightarrow X$ for a subtorus $\boldsymbol{G}_m$ of T. Then X is a finite union of T-orbits of such points and hence a scheme [Hau00, Theorem 1].

For general k, one may immediately reduce to the case that k is a subfield of $\boldsymbol{C}$. The pullback of X to $\boldsymbol{C}$ is a toric variety [Hau00], so a theorem [GS11, Theorem 6.1] implies there exists an étale, representable, surjective morphism $p : [U/\boldsymbol{GL}_n] \to [X_{\boldsymbol{C}}/T]$ where U is a quasi-affine scheme over $\boldsymbol{C}$. Let $L \subset \boldsymbol{C}$ be a subfield of definition of p which is finitely generated over k, giving a morphism $p_L : [U_L/\boldsymbol{GL}_n] \to [X_L/T]$ where U is obtained by pulling back U_L

to $\boldsymbol{C}$. Then $d = \text{tr deg } L/k < \infty$, and p_L remains étale, representable and surjective. Writing $L = k(V)$ for a d-dimensional affine variety V, one may suppose that U_L with its $\boldsymbol{GL}_n$-action is defined over V, realizing p_L as the pullback of a dominant, étale morphism $p_V : [U_V/\boldsymbol{GL}_n] \to [X/T] \times V$ to the generic point of V. After excluding points of U lying in the image of the pullback of the relative inertia, p_V becomes representable. The disjoint union of fibers of p_V over finitely many closed points of V forms an étale, representable, surjective morphism to $[X/T]$. Applying [GS11] in the reverse direction, one deduces that X is a toric algebraic space and hence a scheme. □

Theorem 3.8. *Assume* char $k = 0$*, and let* X *be a Deligne-Mumford stack, smooth and proper over* k*, with an action of* T *which gives a dense, open embedding of* T *in* X*. Then the induced action of any one-dimensional subtorus of* T *induces a Białynicki-Birula decomposition of* X*.*

Proof. By remark 3.3, the action of T on X descends to the coarse moduli space. The theorem now follows from the two previous results. □

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